

# THE SPHERICAL SYSTEMS OF THE WONDERFUL REDUCTIVE SUBGROUPS

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ABSTRACT. We provide the spherical systems of the wonderful reductive subgroups of any reductive group.

## 1. INTRODUCTION

Let  $G$  be a reductive complex algebraic group. An algebraic subgroup  $H$  of  $G$  is called spherical if there exists a Borel subgroup  $B$  of  $G$  such that the subset  $BH$  is open in  $G$ .

The classification of the reductive spherical subgroups of a given reductive group has been known for a long time and is due to M. Krämer [11] and M. Brion [8] (or I.V. Mikityuk [14]).

The classification of all the spherical subgroups of a reductive group is a consequence of the Luna conjecture whose proof has been completed recently with the contribution of several authors, see [13, 15, 5, 2, 12, 4, 3, 6, 7] (and also [9] for a different approach). Such conjecture concerns the classification of special spherical subgroups of a reductive group, called wonderful.

A subgroup  $H$  of  $G$  is wonderful if is the isotropy group of a point in a  $G$ -variety  $X$  with the following properties:  $X$  is complete, non-singular, with an open  $G$ -orbit whose complement is the union of prime  $G$ -divisors with non-empty transversal intersections, and the  $G$ -orbit closures of  $X$  are the partial intersections of these  $G$ -divisors. The conjugacy classes of wonderful subgroups are classified by means of certain combinatorial invariants, called spherical systems.

In the present paper we provide the spherical systems of the wonderful reductive subgroups of any reductive group.

The spherical system of a wonderful subgroup  $H$  of  $G$  encodes in an essential way the information on two types of objects which are naturally associated with  $H$ : the colors and the spherical roots of the homogeneous space  $G/H$ . The set of colors is the set of prime  $B$ -divisors in  $G/H$ . The set of spherical roots is (essentially) the set of equations of the faces of the (simplicial) cone of  $G$ -invariant valuations on the field of rational functions on  $G/H$ .

With a certain effort these data could be computed directly for all spherical subgroups. For example, the colors of  $G/H$  for all the reductive spherical subgroups  $H$  of  $G$  have been explicitly determined, see [11, 1]. For the list presented here we adopted a different, quite indirect approach. On one hand, the abstract spherical systems that can be associated with a reductive wonderful subgroup are classified: this comes from the combinatorial study of all the abstract spherical systems of [3] and a criterion for reductivity due to F. Knop [10]. On the other hand, reductive spherical subgroups are also classified, as mentioned above; therefore, it remains

to find out the right matching between the two lists. The knowledge of the spherical system associated with  $H$  prescribes the dimension of  $H$  and the rank of its character group: this turns out to be enough to uniquely determine the desired correspondence.

The classification of reductive spherical subgroups is a key step in the classification of all spherical subgroups, see for example [12, Proposition 4.2.3]. Furthermore, the list of the spherical systems of the wonderful reductive subgroups, which we present here, may serve as a definitive reference and be of use for the explicit computation of the wonderful subgroup associated with a given spherical system, see [6, Sections 3.1 and 5.2].

## 2. GENERALITIES

**2.1. Notation.** In the following  $H$  will denote a wonderful subgroup of  $G$ . The subgroup  $H$  will be assumed to contain the center of  $G$ , or alternatively  $G$  will be assumed to be semi-simple of adjoint type.

Let us fix a maximal torus  $T$  in  $G$  and a Borel subgroup  $B$  of  $G$  containing  $T$ . The corresponding set of simple roots of the root system of  $(G, T)$  will be denoted by  $S$ .

**2.2. The spherical system associated with  $H$ .** Let  $P_{G/H}$  be the stabilizer of the open  $B$ -orbit of  $G/H$  and denote by  $S_{G/H}^p$  the subset of simple roots corresponding to  $P_{G/H}$ , which is a parabolic subgroup of  $G$  containing  $B$ .

Let  $\Lambda_{G/H}$  be the lattice of  $B$ -weights in  $\mathbb{C}(G/H)$  and  $\mathcal{V}_{G/H} \subset \text{Hom}(\Lambda_{G/H}, \mathbb{Q})$  the cone of  $G$ -invariant valuations on  $\mathbb{C}(G/H)$ . Let  $\Sigma_{G/H} \subset \mathbb{N}S$  be the set of primitive elements in  $\Lambda_{G/H}$  such that

$$\mathcal{V}_{G/H} = \{v \in \text{Hom}(\Lambda_{G/H}, \mathbb{Q}) : \langle v, \sigma \rangle \leq 0 \text{ for all } \sigma \in \Sigma_{G/H}\},$$

it is called the set of spherical roots of  $G/H$ , it is a basis of  $\Lambda_{G/H}$ .

Let  $\mathbf{A}_{G/H}$  be the set of colors that are not stable under a minimal parabolic containing  $B$  and corresponding to a simple root belonging to  $\Sigma_{G/H}$ . There is a  $\mathbb{Z}$ -bilinear pairing  $c_{G/H} : \mathbb{Z}\mathbf{A}_{G/H} \times \mathbb{Z}\Sigma_{G/H} \rightarrow \mathbb{Z}$ , called Cartan pairing, between colors and spherical roots induced by the valuations of  $B$ -stable divisors on  $\mathbb{C}(G/H)^{(B)}$ .

The triple  $\mathcal{S}_{G/H} = (S_{G/H}^p, \Sigma_{G/H}, \mathbf{A}_{G/H})$  is a Luna spherical system in the sense of [13, §2.1]; we will refer to it as the spherical system associated with  $H$ .

**2.3. Information from the spherical system.** The set of colors of  $G/H$  can be retrieved from  $\mathcal{S}_{G/H}$  as the set  $\Delta_{G/H}$  obtained as disjoint union  $\Delta_{G/H}^a \cup \Delta_{G/H}^{2a} \cup \Delta_{G/H}^b$  where:

- $\Delta_{G/H}^a = \mathbf{A}_{G/H}$ ,
- $\Delta_{G/H}^{2a} = \{D_\alpha : \alpha \in S \cap \frac{1}{2}\Sigma_{G/H}\}$ ,
- $\Delta_{G/H}^b = \{D_\alpha : \alpha \in S \setminus (S_{G/H}^p \cup \Sigma_{G/H} \cup \frac{1}{2}\Sigma_{G/H})\} / \sim$ , where  $D_\alpha \sim D_\beta$  if  $\alpha + \beta$  are orthogonal and  $\alpha + \beta \in \Sigma_{G/H}$ .

For all  $\alpha \in S$  set:

$$\Delta_{G/H}(\alpha) = \begin{cases} \emptyset & \text{if } \alpha \in S_{G/H}^p \\ \{D \in \mathbf{A}_{G/H} : c_{G/H}(D, \alpha) = 1\} & \text{if } \alpha \in \Sigma_{G/H} \\ \{D_\alpha\} & \text{otherwise} \end{cases}$$

Then  $\Delta_{G/H}(\alpha)$  corresponds to the subset of colors that are not stable under the minimal parabolic containing  $B$  and corresponding to  $\alpha$ .

The full Cartan pairing of  $G/H$  is the  $\mathbb{Z}$ -bilinear map  $c_{G/H}: \mathbb{Z}\Delta_{G/H} \times \mathbb{Z}\Sigma_{G/H} \rightarrow \mathbb{Z}$  such that:

$$c_{G/H}(D, \sigma) = \begin{cases} c_{G/H}(D, \sigma) & \text{if } D \in \Delta_{G/H}^a \\ \frac{1}{2}\langle \alpha^\vee, \sigma \rangle & \text{if } D = D_\alpha \in \Delta_{G/H}^{2a} \\ \langle \alpha^\vee, \sigma \rangle & \text{if } D = D_\alpha \in \Delta_{G/H}^b \end{cases}$$

The dimension of  $H$  and the rank of its character group can also be read off the spherical system  $\mathcal{S}_{G/H}$ . Indeed, the dimension of  $H$  is equal to

$$\text{card } \Sigma_{G/H} + \dim G/P_{G/H},$$

and the rank of its character group is equal to

$$\text{card } \Delta_{G/H} - \text{card } \Sigma_{G/H}.$$

Finally, the wonderful subgroup  $H$  is reductive if and only if there exists  $\sigma \in \mathbb{N}\Sigma_{G/H}$  such that  $c_{G/H}(D, \sigma) > 0$  for all  $D \in \Delta_{G/H}$ .

**2.4. Direct product.** Let  $H_1$  and  $H_2$  be two subgroups, respectively, of  $G_1$  and  $G_2$ , semi-simple groups of adjoint type. Clearly,  $H_1$  and  $H_2$  are wonderful (reductive) if and only if their direct product  $H_1 \times H_2$  is a wonderful (reductive) subgroup of  $G_1 \times G_2$ . In this case, the spherical system

$$\mathcal{S}_{G_1 \times G_2 / H_1 \times H_2} = ((S_1 \cup S_2)_{G_1 \times G_2 / H_1 \times H_2}^p, \Sigma_{G_1 \times G_2 / H_1 \times H_2}, \mathbf{A}_{G_1 \times G_2 / H_1 \times H_2})$$

is called direct product of the spherical systems  $\mathcal{S}_{G_1/H_1}$  and  $\mathcal{S}_{G_2/H_2}$ . One has:

- $(S_1 \cup S_2)_{G_1 \times G_2 / H_1 \times H_2}^p = (S_1)_{G_1/H_1}^p \cup (S_2)_{G_2/H_2}^p$ ,
- $\Sigma_{G_1 \times G_2 / H_1 \times H_2} = \Sigma_{G_1/H_1} \cup \Sigma_{G_2/H_2}$ ,
- $\mathbf{A}_{G_1 \times G_2 / H_1 \times H_2} = \mathbf{A}_{G_1/H_1} \cup \mathbf{A}_{G_2/H_2}$ , and for all  $D \in \mathbf{A}_{G_i/H_i}$  and  $\sigma \in \Sigma_{G_j/H_j}$  the Cartan pairing  $c_{G_1 \times G_2 / H_1 \times H_2}(D, \sigma)$  equals  $c_{G_i/H_i}(D, \sigma)$  if  $i = j$  or vanishes otherwise.

**2.5. Luna diagrams.** Let us recall how to read the spherical system off its Luna diagram. First, the underlying Dynkin diagram is the Dynkin diagram of  $G$ . The subset of vertices of the Dynkin diagram without circles (neither around nor above nor below) corresponds to the set  $S^p \subseteq S$ . Each spherical root has its own symbol as in Table 2.5.<sup>1</sup> Equivalence classes of circles (two circles are equivalent if they are joined by a line) are in bijective correspondence with colors. A circle corresponds to a color in  $\Delta(\alpha)$  if it is around, above or below the vertex corresponding to the simple root  $\alpha$ . Let us restrict our attention to the set  $\mathbf{A} \subseteq \Delta$ . The subset of simple roots that are spherical roots corresponds to the subset of vertices which have circles above and below, these two circles correspond to the elements of  $\mathbf{A}(\alpha)$ . The Cartan pairing can be retrieved as follows. For all  $D \in \mathbf{A}$  and all  $\sigma \in \Sigma$ ,  $c(D, \sigma)$  equals 1 if and only if  $\sigma$  is a simple root such that  $D \in \mathbf{A}(\sigma)$ . If  $D \in \mathbf{A}(\alpha)$  corresponds to the circle above  $\alpha$  and if  $\sigma$  is a spherical root not orthogonal to  $\alpha$  then  $c(D, \sigma)$  belongs to  $\{1, 0, -1\}$  and equals  $-1$  if and only if there is an arrow starting from the circle corresponding to  $D \in \mathbf{A}(\alpha)$  and pointing toward  $\sigma$ . The

<sup>1</sup>In the table we report only the spherical roots actually occurring in the spherical systems associated with wonderful reductive subgroups, these are all the spherical roots but one.

TABLE 1. Diagrams of spherical roots

diagram	spherical root
	$\alpha_1$
	$2\alpha_1$
	$\alpha_1 + \alpha'_1$
	$\alpha_1 + \dots + \alpha_r$
	$\alpha_1 + 2\alpha_2 + \alpha_3$
	$\alpha_1 + \dots + \alpha_r$
	$2\alpha_1 + \dots + 2\alpha_r$
	$\alpha_1 + 2\alpha_2 + 3\alpha_3$
	$\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{r-1} + \alpha_r$
	$2\alpha_1 + \dots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$
	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$
	$2\alpha_1 + \alpha_2$
	$4\alpha_1 + 2\alpha_2$

rest of the Cartan pairing is uniquely determined by the following identity, for all  $\alpha \in S \cap \Sigma$ , say  $\mathbf{A}(\alpha) = \{D_\alpha^+, D_\alpha^-\}$ , and for all  $\sigma \in \Sigma$

$$c(D_\alpha^+, \sigma) + c(D_\alpha^-, \sigma) = \langle \alpha^\vee, \sigma \rangle.$$

### 3. THE LIST

Here is the list of the wonderful reductive subgroups that cannot be decomposed into a direct product, together with their respective spherical systems.

- (1)  $G \subset G$ , for all simple (adjoint) groups  $G$ :  $S^p = S$ ,  $\Sigma = \emptyset$ ,  $\mathbf{A} = \emptyset$ .
- (2)  $G \subset G \times G$ , for all simple (adjoint) groups  $G$ , where  $G$  is diagonal in  $G \times G$ :  $S^p = \emptyset$ ,  $\Sigma = \{\alpha_i + \alpha'_i : 1 \leq i \leq n\}$ ,  $\mathbf{A} = \emptyset$  (here  $\{\alpha_1, \dots, \alpha_n\}$  and  $\{\alpha'_1, \dots, \alpha'_n\}$  are two (equally ordered) copies of the set of simple roots of  $G$ ).

(3)  $\mathrm{GL}(1) \subset \mathrm{SL}(2)$ :



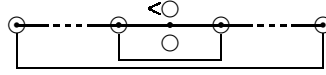
(4)  $\mathrm{N}(\mathrm{GL}(1)) \subset \mathrm{SL}(2)$ :



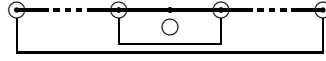
(5)  $\mathrm{GL}(p) \subset \mathrm{SL}(p+1)$ ,  $p \geq 2$ :



(6)  $\mathrm{S}(\mathrm{GL}(p) \times \mathrm{GL}(p)) \subset \mathrm{SL}(2p)$ ,  $p \geq 2$ :



(7)  $\mathrm{N}(\mathrm{S}(\mathrm{GL}(p) \times \mathrm{GL}(p))) \subset \mathrm{SL}(2p)$ ,  $p \geq 2$ :



(8)  $\mathrm{S}(\mathrm{GL}(p) \times \mathrm{GL}(q)) \subset \mathrm{SL}(p+q)$ ,  $1 < p < q$ ,  $\mathrm{card} \Sigma = p$ :



(9)  $\mathrm{N}(\mathrm{SO}(p)) \subset \mathrm{SL}(p)$ ,  $p \geq 3$ :



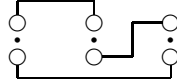
(10)  $\mathrm{N}(\mathrm{Sp}(2p)) \subset \mathrm{SL}(2p)$ ,  $p \geq 2$ :



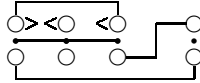
(11)  $\mathrm{N}(\mathrm{GL}(1) \times \mathrm{Sp}(2p)) \subset \mathrm{SL}(2p+1)$ ,  $p \geq 2$ :



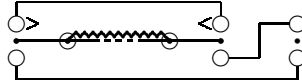
(12)  $\mathrm{N}(\mathrm{SL}(2)) \subset \mathrm{SL}(2) \times \mathrm{SL}(2) \times \mathrm{SL}(2)$ , where  $\mathrm{SL}(2)$  is diagonal in  $\mathrm{SL}(2) \times \mathrm{SL}(2) \times \mathrm{SL}(2)$ :



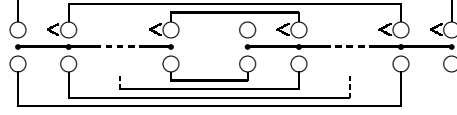
(13)  $\mathrm{GL}(1) \times \mathrm{SL}(2) \times \mathrm{SL}(2) \subset \mathrm{SL}(4) \times \mathrm{SL}(2)$ , where one copy of  $\mathrm{SL}(2)$  is diagonal in  $\mathrm{SL}(4) \times \mathrm{SL}(2)$ :



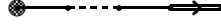
(14)  $\mathrm{GL}(1) \times \mathrm{SL}(2) \times \mathrm{SL}(p) \subset \mathrm{SL}(p+2) \times \mathrm{SL}(2)$ ,  $p \geq 3$ , where  $\mathrm{SL}(2)$  is diagonal in  $\mathrm{SL}(p+2) \times \mathrm{SL}(2)$ :



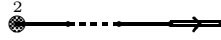
- (15)  $\mathrm{SL}(p) \times \mathrm{GL}(1) \subset \mathrm{SL}(p) \times \mathrm{SL}(p+1)$ ,  $p \geq 2$ , where  $\mathrm{SL}(p)$  is diagonal in  $\mathrm{SL}(p) \times \mathrm{SL}(p+1)$ :



- (16)  $\mathrm{SO}(2p) \subset \mathrm{SO}(2p+1)$ ,  $p \geq 2$ :



- (17)  $\mathrm{N}(\mathrm{SO}(2p)) \subset \mathrm{SO}(2p+1)$ ,  $p \geq 2$ :



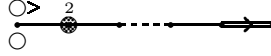
- (18)  $\mathrm{GL}(1) \times \mathrm{SO}(3) \subset \mathrm{SO}(5)$ :



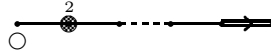
- (19)  $\mathrm{N}(\mathrm{GL}(1) \times \mathrm{SO}(3)) \subset \mathrm{SO}(5)$ :



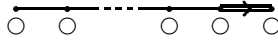
- (20)  $\mathrm{GL}(1) \times \mathrm{SO}(2p-1) \subset \mathrm{SO}(2p+1)$ ,  $p \geq 2$ :



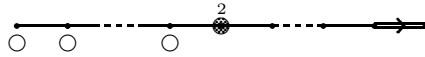
- (21)  $\mathrm{N}(\mathrm{GL}(1) \times \mathrm{SO}(2p-1)) \subset \mathrm{SO}(2p+1)$ ,  $p \geq 2$ :



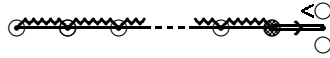
- (22)  $\mathrm{N}(\mathrm{SO}(p) \times \mathrm{SO}(p+1)) \subset \mathrm{SO}(2p+1)$ ,  $p \geq 3$ :



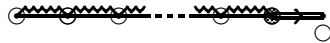
- (23)  $\mathrm{N}(\mathrm{SO}(p) \times \mathrm{SO}(2q+1-p)) \subset \mathrm{SO}(2q+1)$ ,  $2 < p < q$ ,  $\mathrm{card} \Sigma = p$ :



- (24)  $\mathrm{GL}(p) \subset \mathrm{SO}(2p+1)$ ,  $p \geq 2$ :



- (25)  $\mathrm{N}(\mathrm{GL}(p)) \subset \mathrm{SO}(2p+1)$ ,  $p \geq 2$ :



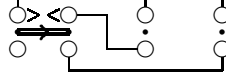
- (26)  $\mathrm{G}_2 \subset \mathrm{SO}(7)$ :



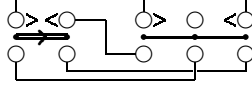
- (27)  $\mathrm{Spin}(7) \subset \mathrm{SO}(9)$ :



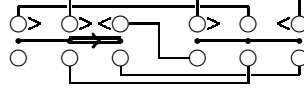
(28)  $N(\mathrm{SO}(4)) \subset \mathrm{SO}(5) \times \mathrm{SO}(4)$ , diagonally:



(29)  $N(\mathrm{SO}(5)) \subset \mathrm{SO}(5) \times \mathrm{SO}(6)$ , diagonally:



(30)  $N(\mathrm{SO}(6)) \subset \mathrm{SO}(7) \times \mathrm{SO}(6)$ , diagonally:



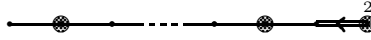
(31)  $\mathrm{SL}(2) \times \mathrm{Sp}(2p) \subset \mathrm{Sp}(2p+2)$ ,  $p \geq 2$ :



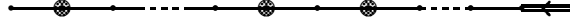
(32)  $\mathrm{Sp}(2p) \times \mathrm{Sp}(2p) \subset \mathrm{Sp}(4p)$ ,  $p \geq 2$ :



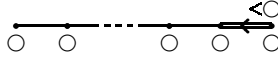
(33)  $N(\mathrm{Sp}(2p) \times \mathrm{Sp}(2p)) \subset \mathrm{Sp}(4p)$ ,  $p \geq 2$ :



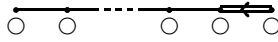
(34)  $\mathrm{Sp}(2p) \times \mathrm{Sp}(2q) \subset \mathrm{Sp}(2p+2q)$ ,  $1 < p < q$ ,  $\mathrm{card} \Sigma = p$ :



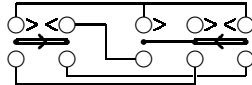
(35)  $\mathrm{GL}(p) \subset \mathrm{Sp}(2p)$ ,  $p \geq 3$ :



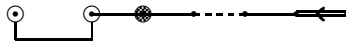
(36)  $N(\mathrm{GL}(p)) \subset \mathrm{Sp}(2p)$ ,  $p \geq 3$ :



(37)  $N(\mathrm{Sp}(4) \times \mathrm{Sp}(2)) \subset \mathrm{Sp}(4) \times \mathrm{Sp}(6)$ , where  $\mathrm{Sp}(4)$  is diagonal in  $\mathrm{Sp}(4) \times \mathrm{Sp}(6)$ :



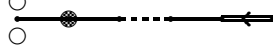
(38)  $N(\mathrm{SL}(2) \times \mathrm{Sp}(2p)) \subset \mathrm{SL}(2) \times \mathrm{Sp}(2p+2)$ ,  $p \geq 2$ , where  $\mathrm{SL}(2)$  is diagonal in  $\mathrm{SL}(2) \times \mathrm{Sp}(2p+2)$ :



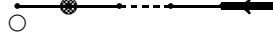
(39)  $N(\mathrm{SL}(2) \times \mathrm{Sp}(2p) \times \mathrm{Sp}(2q)) \subset \mathrm{Sp}(2p+2) \times \mathrm{Sp}(2q+2)$ ,  $p, q \geq 2$ , where  $\mathrm{SL}(2)$  is diagonal in  $\mathrm{Sp}(2p+2) \times \mathrm{Sp}(2q+2)$ :



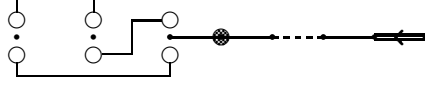
(40)  $GL(1) \times Sp(2p) \subset Sp(2p+2)$ ,  $p \geq 2$ :



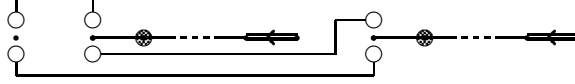
(41)  $N(GL(1) \times Sp(2p)) \subset Sp(2p+2)$ ,  $p \geq 2$ :



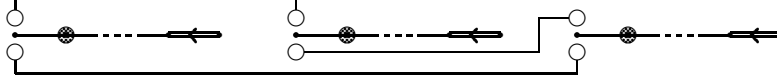
(42)  $N(SL(2) \times Sp(2p)) \subset SL(2) \times SL(2) \times Sp(2p+2)$ ,  $p \geq 2$ , where  $SL(2)$  is diagonal in  $SL(2) \times SL(2) \times Sp(2p+2)$ :



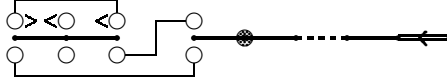
(43)  $N(SL(2) \times Sp(2p) \times Sp(2q)) \subset SL(2) \times Sp(2p+2) \times Sp(2q+2)$ ,  $p, q \geq 2$ , where  $SL(2)$  is diagonal in  $SL(2) \times Sp(2p+2) \times Sp(2q+2)$ :



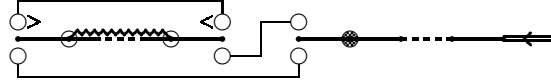
(44)  $N(SL(2) \times Sp(2p) \times Sp(2q) \times Sp(2r)) \subset Sp(2p+2) \times Sp(2q+2) \times Sp(2r+2)$ ,  $p, q, r \geq 2$ , where  $SL(2)$  is diagonal in  $Sp(2p+2) \times Sp(2q+2) \times Sp(2r+2)$ :



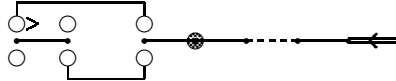
(45)  $GL(1) \times SL(2) \times SL(2) \times Sp(2q) \subset SL(4) \times Sp(2q+2)$ ,  $q \geq 2$ , where one copy of  $SL(2)$  is diagonal in  $SL(4) \times Sp(2q+2)$ :



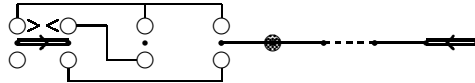
(46)  $GL(1) \times SL(2) \times SL(p) \times Sp(2q) \subset SL(p+2) \times Sp(2q+2)$ ,  $p \geq 3$ ,  $q \geq 2$ , where  $SL(2)$  is diagonal in  $SL(p+2) \times Sp(2q+2)$ :



(47)  $GL(1) \times SL(2) \times Sp(2p) \subset SL(3) \times Sp(2p+2)$ ,  $p \geq 2$ , where  $SL(2)$  is diagonal in  $SL(3) \times Sp(2p+2)$ :

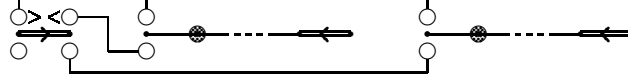


(48)  $N(SL(2) \times SL(2) \times Sp(2p)) \subset Sp(4) \times SL(2) \times Sp(2p+2)$ ,  $p \geq 2$ , where one copy of  $SL(2)$  is diagonal in  $Sp(4) \times SL(2) \times Sp(2p+2)$ :

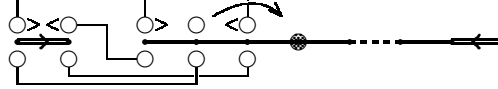




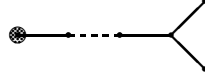
- (49)  $N(\mathrm{SL}(2) \times \mathrm{SL}(2) \times \mathrm{Sp}(2p) \times \mathrm{Sp}(2q)) \subset \mathrm{Sp}(4) \times \mathrm{Sp}(2p+2) \times \mathrm{Sp}(2q+2)$ ,  $p, q \geq 2$ , where one copy of  $\mathrm{SL}(2)$  is diagonal in  $\mathrm{Sp}(4) \times \mathrm{Sp}(2p+2) \times \mathrm{Sp}(2q+2)$ :



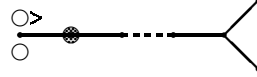
- (50)  $N(\mathrm{Sp}(4) \times \mathrm{Sp}(2p)) \subset \mathrm{Sp}(4) \times \mathrm{Sp}(2p+4)$ ,  $p \geq 2$ , where  $\mathrm{Sp}(4)$  is diagonal in  $\mathrm{Sp}(4) \times \mathrm{Sp}(2p+4)$ :



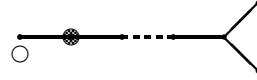
- (51)  $N(\mathrm{SO}(2p-1)) \subset \mathrm{SO}(2p)$ ,  $p \geq 4$ :



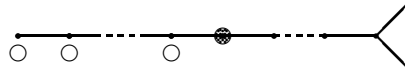
- (52)  $\mathrm{GL}(1) \times \mathrm{SO}(2p) \subset \mathrm{SO}(2p+2)$ ,  $p \geq 3$ :



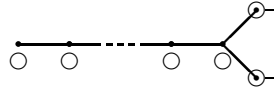
- (53)  $N(\mathrm{GL}(1) \times \mathrm{SO}(2p)) \subset \mathrm{SO}(2p+2)$ ,  $p \geq 3$ :



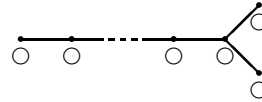
- (54)  $N(\mathrm{SO}(p) \times \mathrm{SO}(2q-p)) \subset \mathrm{SO}(2q)$ ,  $3 \leq p \leq q-2$ ,  $\mathrm{card} \Sigma = p$ :



- (55)  $N(\mathrm{SO}(p) \times \mathrm{SO}(p+2)) \subset \mathrm{SO}(2p+2)$ ,  $p \geq 3$ :



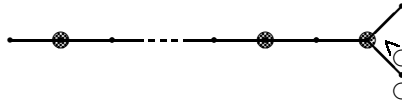
- (56)  $N(\mathrm{SO}(p) \times \mathrm{SO}(p)) \subset \mathrm{SO}(2p)$ ,  $p \geq 4$ :



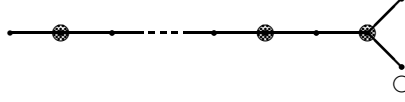
- (57)  $\mathrm{GL}(2p+1) \subset \mathrm{SO}(4p+2)$ ,  $p \geq 2$ :



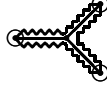
- (58)  $\mathrm{GL}(2p) \subset \mathrm{SO}(4p)$ ,  $p \geq 2$ :



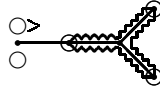
(59)  $N(\mathrm{GL}(2p)) \subset \mathrm{SO}(4p)$ ,  $p \geq 2$ :



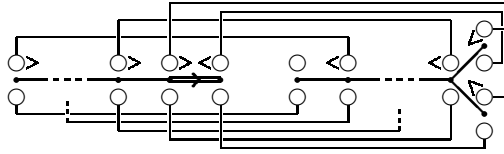
(60)  $G_2 \subset \mathrm{SO}(8)$ :



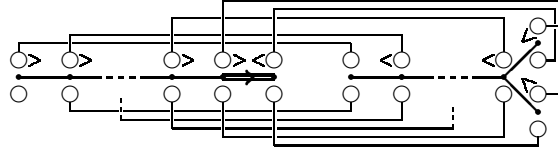
(61)  $\mathrm{GL}(1) \times \mathrm{Spin}(7) \subset \mathrm{SO}(10)$ :



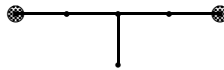
(62)  $N(\mathrm{SO}(2p-1)) \subset \mathrm{SO}(2p-1) \times \mathrm{SO}(2p)$ ,  $p \geq 4$ , diagonally:



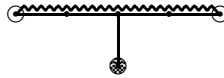
(63)  $N(\mathrm{SO}(2p)) \subset \mathrm{SO}(2p+1) \times \mathrm{SO}(2p)$ ,  $p \geq 4$ , diagonally:



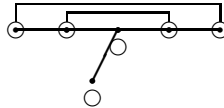
(64)  $N(F_4) \subset E_6$ :



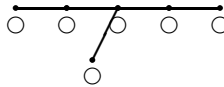
(65)  $\mathrm{GL}(1) \times \mathrm{Spin}(10) \subset E_6$ :



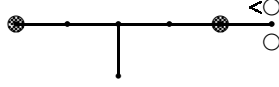
(66)  $N(\mathrm{SL}(6) \times \mathrm{SL}(2)) \subset E_6$ :



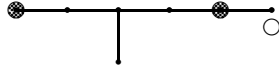
(67)  $N(\mathrm{Sp}(8)) \subset E_6$ :



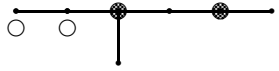
(68)  $\mathrm{GL}(1) \times \mathrm{E}_6 \subset \mathrm{E}_7$ :



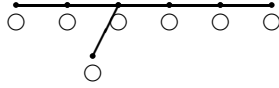
(69)  $\mathrm{N}(\mathrm{GL}(1) \times \mathrm{E}_6) \subset \mathrm{E}_7$ :



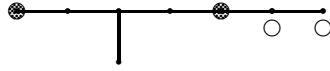
(70)  $\mathrm{N}(\mathrm{Spin}(12) \times \mathrm{SL}(2)) \subset \mathrm{E}_7$ :



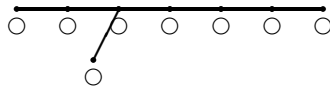
(71)  $\mathrm{N}(\mathrm{SL}(8)) \subset \mathrm{E}_7$ :



(72)  $\mathrm{N}(\mathrm{SL}(2) \times \mathrm{E}_7) \subset \mathrm{E}_8$ :



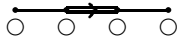
(73)  $\mathrm{N}(\mathrm{Spin}(16)) \subset \mathrm{E}_8$ :



(74)  $\mathrm{Spin}(9) \subset \mathrm{F}_4$ :



(75)  $\mathrm{Sp}(6) \times \mathrm{SL}(2) \subset \mathrm{F}_4$ :



(76)  $\mathrm{SL}(3) \subset \mathrm{G}_2$ :



(77)  $\mathrm{N}(\mathrm{SL}(3)) \subset \mathrm{G}_2$ :



(78)  $\mathrm{SL}(2) \times \mathrm{SL}(2) \subset \mathrm{G}_2$ :



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